The well-posedness for the direct problem and inverse problems for time-fractional partial differential equations: some fundamental studies

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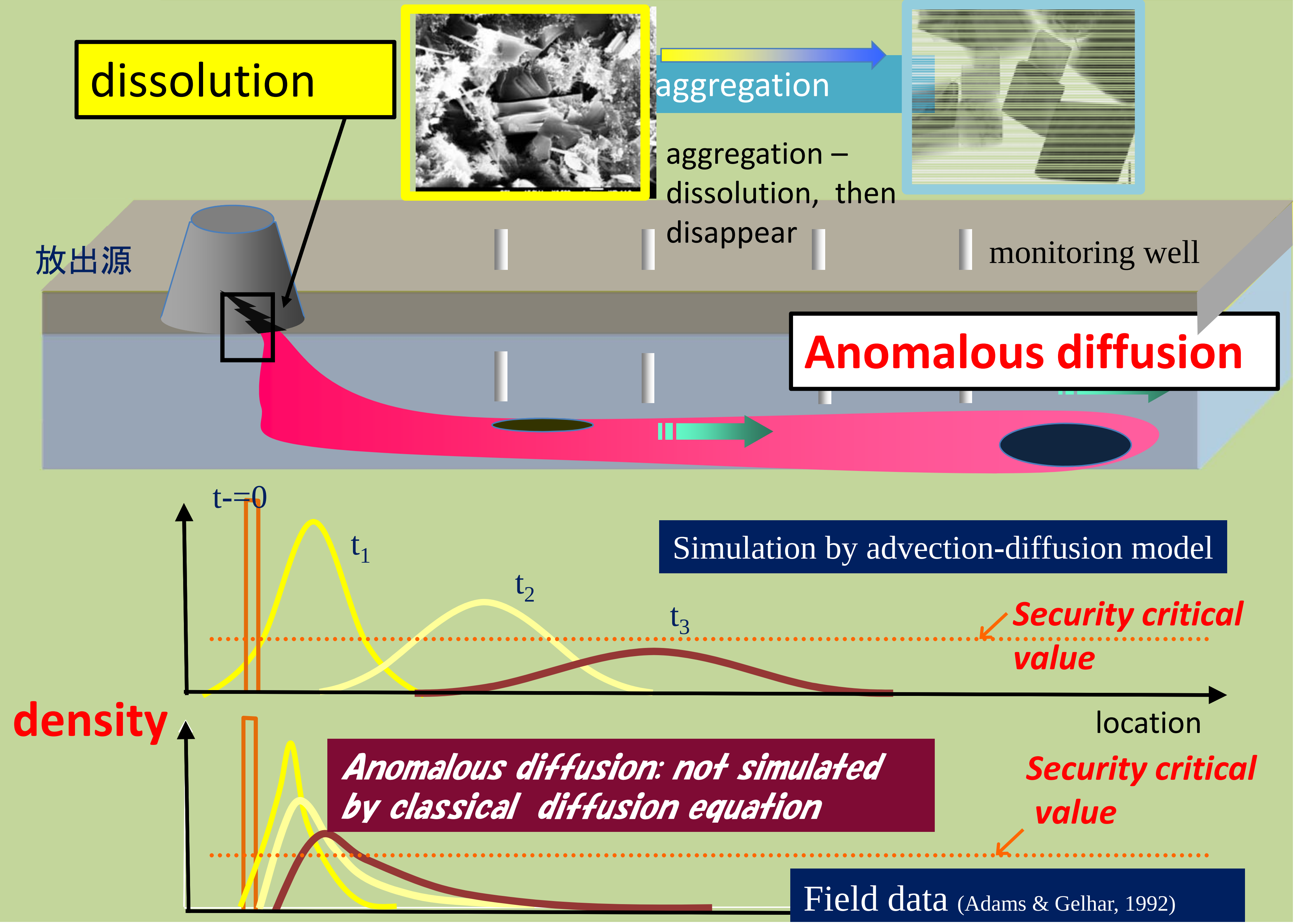
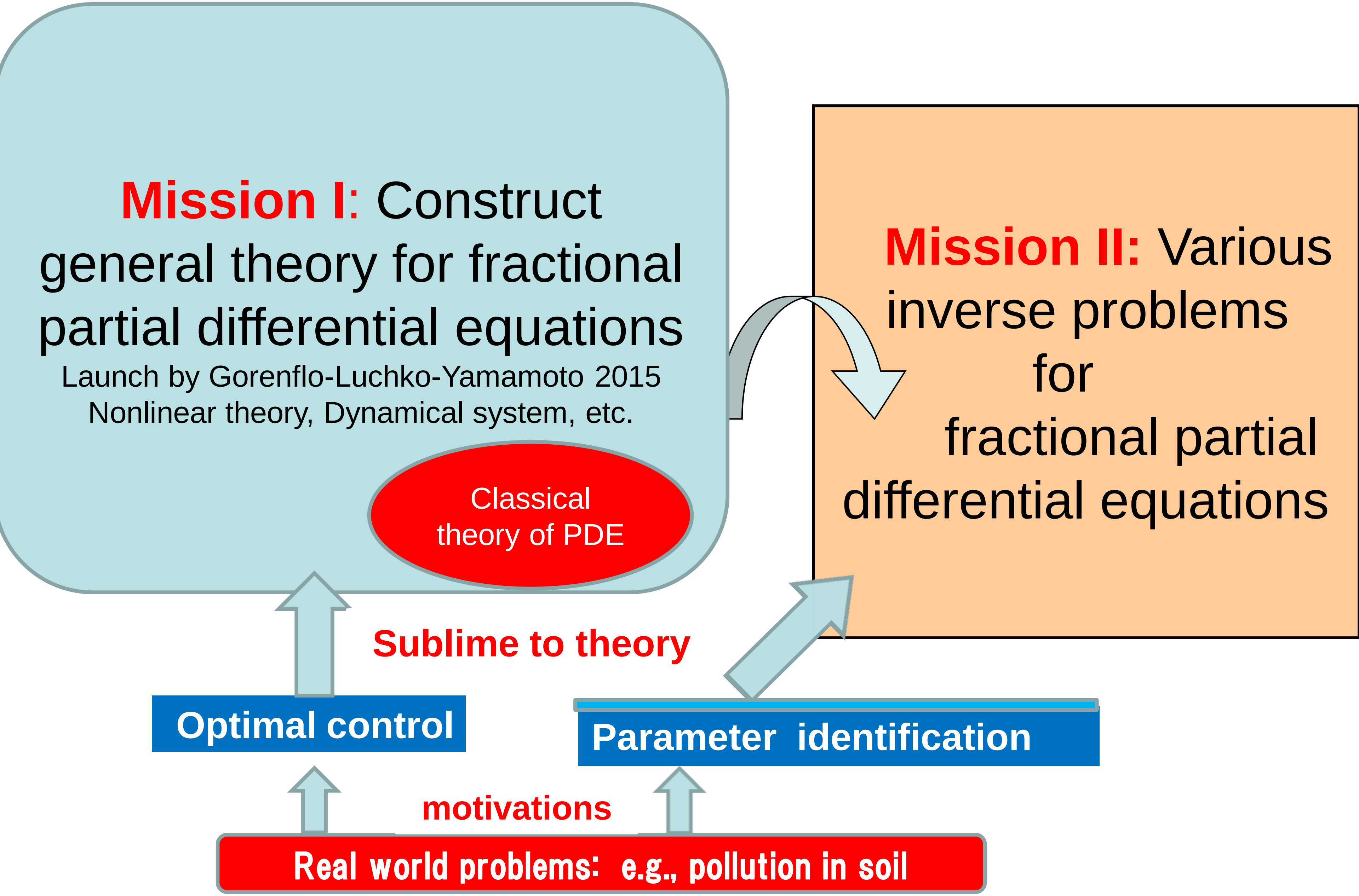
23 July 2020

From ultraslow dissolution to anomalous diffusion



Grand Research Plan



Part I: Direct problems for generalized time-fractional partial differential equations

Part II: Recent results for inverse problems

Contents of Part I

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- §4. Extremum principle - comparison principles
- §5. Coercivity of $\partial_{t,\gamma}^{\alpha}$
- §6. Initial-boundary value problem for time-dependent coefficients

Final purpose: unique existence of weak and strong solutions to initial boundary value problem

$$\begin{cases} \partial_{t,\gamma}^{\alpha} u & = \sum_{i,j=1}^{n} \partial_i(a_{ij}(x,t)\partial_j u) \\ & + \sum_{j=1}^{n} b_j(x,t)\partial_j u + c(x,t)u + F(x,t), \\ & u|_{\partial\Omega} = 0, \quad u(\cdot,0) = a. \end{cases}$$

$\partial_{t,\gamma}^{\alpha}$: generalized Caputo derivative, $0 < \alpha < 1$:

$$\partial_{t,\gamma}^{\alpha} v(t) := \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \gamma(t-s) \frac{dv}{ds}(s) ds$$

Glance at the results:

Similar results to $\gamma \equiv 1$ (Caputo derivative) for

- extremum principle - comparison principles
- coercivity of $\partial_{t,\gamma}^{\alpha}$
- Weak solutions to initial-boundary value problem for time-dependent coefficients

Same treatment for $\alpha > 1$ except for extremum principle

§1. Introduction

Caputo derivative: $\partial_t^\alpha v(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \partial_s v(s) ds$, $0 < \alpha < 1$
for $v \in W^{1,1}(0,T)$ ($\Longleftarrow v, v' \in L^1(0,T)$))
Which class of v should we consider? $W^{1,1}(0,T)$ is too narrow!

Example. $\partial_t^\alpha v(t) = f(t) \in L^2(0,T)$ and $v(0) = 1$ with $0 < \alpha < \frac{1}{2}$: no solutions!!!
$\Longleftarrow$ for $f(t) := t^{\delta - \frac{1}{2}} \in L^2(0,T)$ and small $\delta > 0$ we should have $v(t) = c_0 t^{\alpha + \delta - \frac{1}{2}}$

For $f \in L^\infty(0,T)$, solution exists.

$\Longrightarrow$ The domain for ∂_t^α should be wider than $W^{1,1}(0,T)$ in order to admit $f \in L^2(0,T)$.

§2. Generalized Caputo derivative in Sobolev spaces

Generalization of the theory for Caputo derivative to

$$\partial_{t,\gamma}^{\alpha} v(t) := \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \gamma(t-s) \frac{dv}{ds}(s)\,ds, \quad 0 < \alpha < 1$$

Assume

$$\begin{cases} \gamma \in W^{1,\kappa}(0,T), \exists \kappa > 1, \quad \gamma(0) \neq 0, \\ \sup_{0<\xi<T} \left| \xi^{\beta} \frac{\partial \gamma}{\partial \xi}(\xi) \right| < \infty \quad \exists \beta \in (0,1) \end{cases}$$

$\Longleftarrow \gamma \in C^1[0,T]$ and $\gamma(0) \neq 0$ are enough.

$\gamma \equiv 1 \Longrightarrow$ classical Caputo derivative

Theoretical topics of my talk

- Isomorphism of $\partial_{t,\gamma}^{\alpha}$ in the Sobolev spaces
- Maximum principle: extremum principle
- Coercivity of $\partial_{t,\gamma}^{\alpha}$
- IBVP (initial boundary value problem) for $\partial_{t,\gamma}^{\alpha} u = A(t)u + F$

Existing works from other viewpoint

$$\partial^{\alpha}_{t,\gamma} v(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \gamma(t-s) \frac{dv}{ds}(s) ds$$

Set $g(t) := \frac{\gamma(t)}{\Gamma(1-\alpha) t^{\alpha}}$.

Kochubei (2011), Luchko-Yamamoto (2016)

Their class of γ requires

- $(Lg)(p) := \int_0^{\infty} e^{-pt} g(t) dt$ exists for $p > 0$.
- $\lim_{p\to\infty} (Lg)(p) = 0$, $\lim_{p\to\infty} p(Lg)(p) = \infty$
- $\lim_{p\to 0} (Lg)(p) = \infty$, $\lim_{p\to 0} p(Lg)(p) = 0$, etc.

$\gamma \equiv 1$ satisfies all!

The class of γ guarantees complete monotonicity of solution ψ_λ with $\lambda > 0$:

$$\partial_{t,\gamma}^{\alpha} \psi(t) = -\lambda\psi(t), \quad \psi(0) = 1$$

We say ψ complete monotonicity if

$$(-1)^n \frac{d^n \psi}{dt^n}(t) \geq 0, \quad t > 0,\ n = 0, 1, 2, 3, \ldots$$

$\gamma \equiv 1 \Longrightarrow \psi_\lambda(t) = E_{\alpha,1}(-\lambda t^\alpha)$: complete monotone

Our class for γ is different and aims more at PDE-consideration e.g., for Sobolev regularity!

§3. Definition of $\partial_{t,\gamma}^{\alpha}$ in Sobolev spaces

$\Longleftarrow$ Gorenflo-Luchko-Yamamoto (2015), Kubica-Ryszewska-Yamamoto (2020)

$$H_\alpha(0,T) := \begin{cases} H^\alpha(0,T), \quad 0 < \alpha < \frac{1}{2}, \\ \left\{ v \in H^{\frac{1}{2}}(0,T);\ \int_0^T \frac{|v(t)|^2}{t} dt < \infty \right\}, \\ \{u \in H^\alpha(0,T);\ u(0) = 0\}, \quad \frac{1}{2} < \alpha \le 1 \end{cases}$$

$$\|u\|_{H_\alpha(0,T)} := \begin{cases} \|u\|_{H^\alpha(0,T)}, & \alpha \ne \frac{1}{2}, \\ \left(\|u\|^2_{H^{\frac{1}{2}}(0,T)} + \int_0^T \frac{|u(t)|^2}{t} dt \right)^{1/2}, & \alpha = \frac{1}{2}. \end{cases}$$

Recall $\partial_{t,\gamma}^{\alpha} v(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \gamma(t-s) \frac{dv}{ds}(s)ds$, $0 < \alpha < 1$ for $v \in W^{1,1}(0,T)$

Assume

$$\begin{cases} \gamma \in W^{1,\kappa}(0,T) \quad \gamma(0) \neq 0, \\ \sup_{0<\xi<T} \left| \xi^{\beta} \frac{\partial \gamma}{\partial \xi}(\xi) \right| < \infty \quad \text{with some } \kappa > 1 \text{ and } \beta \in (0,1). \end{cases}$$

Theorem 1 (Yamamoto 2020).

Unique extension $\overline{\partial_{t,\gamma}^{\alpha}}$ *of* $\partial_{t,\gamma}^{\alpha}$ *exists such that*

(i) $\overline{\partial_{t,\gamma}^{\alpha}} v$ *is defined for* $v \in H_{\alpha}(0,T)$

(ii) $\|\partial_{t,\gamma}^{\alpha} v\|_{L^2(0,T)} \sim \|v\|_{H_{\alpha}(0,T)}$

Henceforth we write $\overline{\partial_{t,\gamma}^{\alpha}} = \partial_{t,\gamma}^{\alpha}$

Essence: $\frac{\gamma(t) t^{-\alpha}}{\Gamma(1-\alpha)} = \frac{\gamma(0) t^{-\alpha}}{\Gamma(1-\alpha)} + o(t^{-\alpha})$ = Caputo derivative + perturbation

Representation of $\partial_{t,\gamma}^\alpha$:

(Gorenflo-Luchko-Yamamoto 2015) $\Longrightarrow$

$J^\alpha v(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s)ds : L^2(0,T) \longrightarrow H_\alpha(0,T)$ is bijective and isomorphism

Set

$$\partial_t^\alpha = (J^\alpha)^{-1}, \quad q(s) := \int_0^1 (1-\eta)^{\alpha-1}\eta^{-\alpha}\gamma(s\eta)d\eta, \quad (Ku)(t) := -\int_0^t \frac{dq}{ds}(t-s)u(s)ds$$

$$\Longrightarrow (\partial_{t,\gamma}^\alpha v)(t) = \partial_t^\alpha \left(\gamma(0) - \frac{\sin \pi\alpha}{\pi} K\right) v(t), \quad 0 < t < T,\ v \in H_\alpha(0,T)$$

Here $K : L^2(0,T) \longrightarrow L^2(0,T)$; compact by $\left|\frac{dq}{d\eta}(s)\right| \le Cs^{-\beta}$ (weak Hilbert-Schmidt opertor)

$\gamma(0) - \frac{\sin \pi\alpha}{\pi} K : L^2(0,T) \longrightarrow L^2(0,T)$: isomorphism

Application to fractional ordinary differential equations

Let $r \in L^\infty(0,T)$ and $f \in L^2(0,T)$,

$$\partial_{t,\gamma}^\alpha (v(t) - a) = r(t)v(t) + f(t), \quad t > 0, \quad v - a \in H_\alpha(0,T)$$

Remark: $v - a \in H_\alpha(0,T) \Longrightarrow v(0) = a$ for $\frac{1}{2} < \alpha$
because $v - a \in H_\alpha(0,T) \Longrightarrow v(0) - a = 0$

Unique existence of solution: Theorem 1 $\Longrightarrow$
$L_\alpha := (\partial_{t,\gamma}^\alpha)^{-1} : L^2(0,T) \longrightarrow H_\alpha(0,T)$ is isomorphism

$$\partial_{t,\gamma}^\alpha (v(t) - a) = r(t)v(t) + f(t), \quad t > 0, \quad v - a \in H_\alpha(0,T)$$

$$\Longleftrightarrow v = L_\alpha(rv) + a + L_\alpha f$$

L_α is compact in $L^2(0,T)$! Fredholm alternative $\Longrightarrow$ unique existence

Proof of uniqueness: ($\Longrightarrow$ conclusion)

Let $u = L_\alpha(ru) \Longrightarrow$

$$u(t) = -\frac{\sin \pi\alpha}{\pi\gamma(0)} \int_0^t \frac{dq}{ds}(t-s)u(s)ds + \frac{1}{\gamma(0)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}(ru)(s)ds$$

$\Longrightarrow$

$$|u(t)| \leq C \int_0^t (t-s)^{-\beta}|u(s)|ds + C \int_0^t (t-s)^{\alpha-1}|u(s)|ds$$

Gronwall inequality $\Longrightarrow u \equiv 0$

§4. Extremum principle

Stronger assumption:
$\gamma \in C^1[0,\infty)$, $\gamma(t) > 0$, $t\gamma'(t) - \alpha\gamma(t) \le 0$ for $t \ge 0$.

Theorem 2 (Y. 2020).
Let $v \in C^1(0,T]$ *and* $v' \in L^1(0,T)$ *and attain minimum at* $0 < t_0 \le T$. *Then*

$$\partial_{t,\gamma}^{\alpha} v(t_0) \le 0$$

Remark: maxmimum $\Longrightarrow \partial_{t,\gamma}^{\alpha} v(t_0) \ge 0$ (replace v by $-v$).

Proof. Similar to Luchko (2009).

Application: Let $c(x,t) < 0$ and

$$\partial_{t,\gamma}^{\alpha} u(x,t) = \Delta u + c(x,t)u(x,t), \quad x \in \Omega,\ t > 0$$

If $u|_{\partial\Omega\times(0,T)} \geq 0$ and $u(\cdot,0) \geq 0$, then

$$u \geq 0 \quad \text{on } \overline{\Omega} \times [0,T]$$

under some regularity.

We can relax regularity and remove $c < 0$.

§5. Coercivity

Trivial for $\alpha = 1$: $\int_0^T v(t)\partial_t v(t)dt \ge 0$ if $v(0) = 0$.

Theorem 4 (Y. 2020).

Let

$$\begin{cases} \gamma \in C^1[0,T], \\ \gamma(t) > 0, \quad t\gamma'(t) - \alpha\gamma(t) \le 0, \quad 0 \le t \le T \end{cases}$$

(i) $v(t)\partial_{t,\gamma}^{\alpha} v(t) \ge \frac{1}{2}(\partial_{t,\gamma}^{\alpha}|v|^2)(t),\ 0 \le t \le T$

for $v \in C^1[0,T]$ *with* $v(0) = 0$.

$$\text{(ii)} \quad \int_0^t (t-s)^{\alpha-1} v(s)\partial_{t,\gamma}^{\alpha} v(s)ds \ge \exists c_0|v(t)|^2$$

for $0 < t < T$ *and* $v \in H_\alpha(0,T)$.

Application of coercivity: a priori estimate

$$\begin{cases} \partial_{t,\gamma}^{\alpha}(u-a) = \Delta u \quad \text{in } \Omega \times (0,T), \\ u|_{\partial\Omega} = 0, \quad u(\cdot,0) = a \in H^2(\Omega) \cap H_0^1(\Omega) \end{cases}$$

$$\Longrightarrow \partial_{t,\gamma}^{\alpha}(u-a) = \Delta(u-a) + \Delta a \Longrightarrow$$

$$\int_\Omega \partial_{t,\gamma}^{\alpha}(u-a)(x,s)(u-a)(x,s)dx + \int_\Omega |\nabla(u-a)(x,s)|^2 dx = -\int_\Omega \nabla a \cdot \nabla(u-a)(x,s)dx$$

$\int_0^t (t-s)^{\alpha-1} \cdots ds$: order-preserving

$$c_1 \|(u-a)(\cdot,t)\|^2_{L^2(\Omega)} + \int_0^t (t-s)^{\alpha-1} \|\nabla(u-a)(\cdot,s)\|^2_{L^2(\Omega)} ds$$

$$\le C_\varepsilon \int_0^t (t-s)^{\alpha-1} ds \|\nabla a\|^2_{L^2(\Omega)} + \varepsilon \int_0^t (t-s)^{\alpha-1} \|\nabla(u-a)(\cdot,s)\|^2_{L^2(\Omega)} ds$$

with $\varepsilon << 1$.

$\Longrightarrow$

$$\|(u-a)(\cdot,t)\|^2_{L^2(\Omega)} + \int_0^t (t-s)^{\alpha-1}\|\nabla(u-a)(\cdot,s)\|^2_{L^2(\Omega)}\,ds \le C\|\nabla a\|^2_{L^2(\Omega)}$$

$\Longrightarrow$ A priori estimate:

$$\|u-a\|_{L^\infty(0,T;L^2(\Omega))} + \|\nabla(u-a)\|^2_{L^2(0,T;L^2(\Omega))} \le C\|\nabla a\|^2_{L^2(\Omega)}$$

§6. Initial boundary value problems

We assume

$$\begin{cases} \gamma \in C^1[0,T], \\ \gamma(t) > 0, \\ t\gamma'(t) - \alpha\gamma(t) \le 0, \quad 0 \le t \le T \end{cases}$$

$$
\begin{cases}
\partial_{t,\gamma}^{\alpha}(u-a)(x,t)+A(t)u(x,t)=F \quad \text{in } H^{-1}(\Omega), \\
u(\cdot,t)\in H_0^1(\Omega), \\
u(x,\cdot)-a(x)\in H_\alpha(0,T), \quad x\in\Omega.
\end{cases}
$$

Here $\Omega \subset \mathbb{R}^d$ smooth domain, $H^{-1}(\Omega)=(H_0^1(\Omega))'$,

$-A(t)u=\sum_{i,j=1}^{n}\partial_i(a_{ij}(x,t)\partial_j u)+\sum_{j=1}^{n} b_j(x,t)\partial_j u+c(x,t)u$,

where $a_{ij}=a_{ji}, b_j, c\in C^2([0,T];C^1(\overline{\Omega}))$.

Theorem 5 (weak solution).

Let $F \in L^2(0,T;H^{-1}(\Omega))$ *and* $a \in L^2(\Omega)$.

(i) $\exists$ *1* $u \in L^2(0,T;H_0^1(\Omega))$ *such that*

$$\|u-a\|_{H_\alpha(0,T;H^{-1}(\Omega))} + \|\nabla u\|_{L^2(0,T;L^2(\Omega))}$$
$$\leq C(\|a\|_{L^2(\Omega)} + \|F\|_{L^2(0,T;H^{-1}(\Omega))})$$

(ii) (continuity at $t=0$*).*

Let $F \in L^p(0,T;H^{-1}(\Omega))$ *with* $p > \frac{2}{\alpha}$, $a \in H_0^1(\Omega)$. *Then*

$$\|u-a\|_{L^\infty(0,t;L^2(\Omega))} \leq C(t^{\alpha/2}\|a\|_{H_0^1(\Omega)} + t^\theta \|F\|_{L^p(0,T;H^{-1}(\Omega))})$$

with $\theta = \frac{p\alpha - 2}{p-2} > 0$.

Theorem 6 (strong solution).

Let $F \in L^2(0,T;L^2(\Omega))$ *and* $a \in H_0^1(\Omega)$.

$\exists$ *1* $u \in L^2(0,T;H^2(\Omega))$ *such that*

$$\|u-a\|_{H_\alpha(0,T;L^2(\Omega))} + \|u\|_{L^2(0,T;H^2(\Omega))}$$
$$\le C(\|a\|_{H_0^1(\Omega)} + \|F\|_{L^2(0,T;L^2(\Omega))})$$

Key to proofs: Galerkin method
Kubica-Ryszewska-Yamamoto 2020 for classical Caputo derivative

- Construct solutions of finite dimensional approximating systems
- Coercivity $\Longrightarrow$ uniform a priori estimate of approximate solutions
- Weak convergent subsequence to the solution

Part II. Inverse Problems

Contents of Part II:
Three kinds of inverse problems for fractional partial differential equations

Messages:
We have many interesting and important inverse problems.

Contents of Part II

§1. Introduction

$\Omega \subset \mathbb{R}^d$ with smooth boundary $\partial\Omega$

$$\partial_t^\alpha u(x,t) = \sum_{i,j=1}^{d} \partial_i(a_{ij}(x)\partial_j u(x,t)) + c(x)u$$

Forward problem:

Solve initial boundary value problem $\Longrightarrow$ prediction

Inverse problems:

How to determine α, a_{ij}, initial, boundary value, shape of Ω. $\Longrightarrow$

modelling, basement for forward problems

Variety of inverse problems!

Mathematical issues for inverse problems:
uniqueness, stability

varieties of inverse problems × varieties
of fractional equations as models
$\Longrightarrow$ **Varieties^{100}**

Many interesting inverse problems for fractional
differential equations!!

§2. Determination of fractional orders

$$
u_{\alpha,\beta}\begin{cases}
\partial_t^\alpha u = -(-\Delta)^\beta u, & x \in \Omega,\ t > 0, \\
u|_{\partial\Omega} = 0, & t > 0, \\
\begin{cases}
u(x,0) = a(x), & x \in \Omega \quad \text{if } 0 < \alpha < 1, \\
u(x,0) = a(x), & \partial_t u(x,0) = 0 \quad \text{if } 1 < \alpha < 2.
\end{cases}
\end{cases}
$$

Inverse problem: determine $\alpha \in (0,2)$ and $\beta \in (0,1)$ by $u(x_0,t)$, $0 < t < T$ with fixed $x_0 \in \Omega$.

Remark.

- Caputo derivative:

$$\partial_t^{\alpha} v(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} \frac{d^n v}{ds^n}(s)\,ds, \quad n-1 < \alpha < n.$$

- We can replace $-\Delta$ by $\sum_{i,j=1}^{d} \partial_i (a_{ij}(x)\partial_j) + c(x)$.
  Maybe we can treat non-symmetric operators.

About forward problems: a monograph
Kubica-Ryszewska-Yamamoto (Springer, 2020)

Remark. $(-\Delta)^\beta$: fractional power of $-\Delta$

$\Longleftarrow$ spatial derivative of order 2β

$\alpha, \beta \Longrightarrow$ important physical parameters

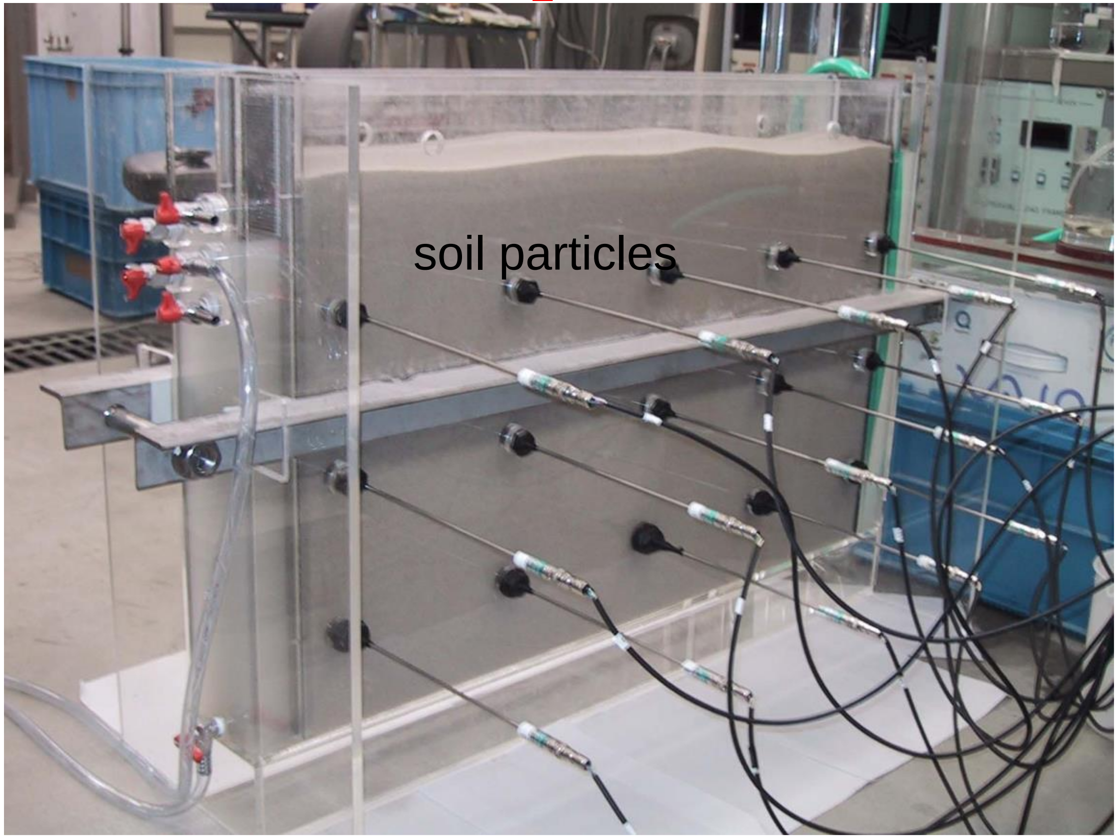
Inverse problem.

Let $x_0 \in \Omega$ be fixed. $u(x_0, t)$, $0 < t < T \Longrightarrow$

$\alpha \in ((0, 2) \setminus \{1\})$ and $\beta \in (0, 1)$.

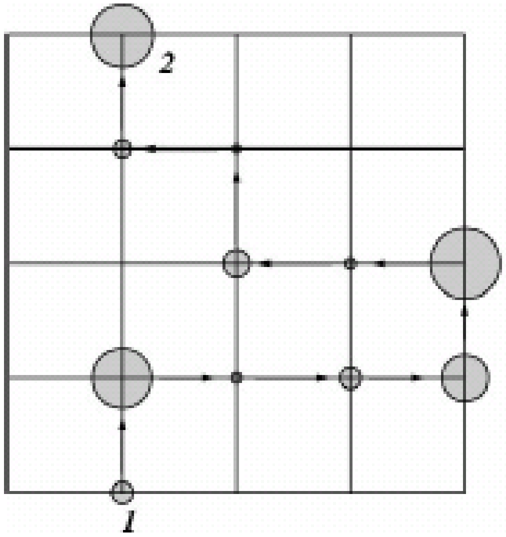
Uniqueness $\Longleftarrow$ How much information data have?

Determination of fractional orders at laboratory based on the micro-model



Prof. Y. Hatano (Tsukuba University)

Comparison of laboratory data with numerical results by Monte Carlo method



Ranom walk

Normal diffusion by Fick's law

$$\langle x^2 \rangle \propto t$$

α=1

Continuous time random walk (micro-model)

$$\langle x^2 \rangle \propto t^{\alpha} \qquad 0 < \alpha < 1$$

Related works on determination of orders

- Hatano, Nakagawa, Wang and Yamamoto 2013
- Li and Yamamoto 2015: uniqueness for mutiterm cases
- Yu, Jing and Qi 2015
- Janno 2016: unique existence
- Janno and Kinash 2018
- Krasnoschok, Pereverzyev, Siryk and Vasylyeva 2019
- Ashurov and Umarov 2020: arXiv:2005.13468v1

Other examples of data:

$$\int_{\Omega} u(x,t)\rho(x)dx, \quad 0 < t < T : \ \rho\text{: weight}$$

$$\nabla u \cdot \nu(x_0, t), \quad 0 < t < T$$

$\nu(x)$: unit outward normal vector

Preparations

$0 < \lambda_1 < \lambda_2 < \cdots \quad \longrightarrow \infty$: set of eigenvalues of $-\Delta$ with $u|_{\partial\Omega} = 0$.

$-\Delta\varphi_{kj} = \lambda_k\varphi_{kj}$, $1 \le j \le m_k$, m_k: multiplicity of λ_k

φ_{kj}, $1 \le j \le m_k$: orthonormal basis

$(\varphi_{kj}, \varphi_{ij}) := \int_\Omega \varphi_{kj}(x)\varphi_{ki}(x)dx = 1$ if $i = j$, $= 0$ if $i \ne j$

$\lambda_k, \{\varphi_{kj}\}_{1\le j\le m_k}$: eigensystem of $-\Delta$ with $u|_{\partial\Omega} = 0$

Fractional power of $-\Delta$:

$$(-\Delta)^{\beta} v = \sum_{k=1}^{\infty} \lambda_k^{\beta} \sum_{j=1}^{m_k} (v, \varphi_{kj}) \varphi_{kj},$$

$v \in D((-\Delta)^{\beta}) \subset H^{2\beta}(\Omega)$: Sobolev-Slobodeckij space

$$(-\Delta)^{-\beta} a = \frac{\sin \pi\beta}{\pi} \int_0^{\infty} \eta^{-\beta} (-\Delta + \eta)^{-1} a \, d\eta$$

in $L^2(\Omega)$ with $0 < \beta < 1$ (e.g., Pazy).

$$E_{\alpha,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)} :$$ Mittag-Leffler function

Theorem 1 (uniqueness).

Let $a \in H_0^2(\Omega)$ *if* $d = 1, 2, 3$ *(*$a \in H_0^{2\gamma}(\Omega)$ *with* $\gamma > \frac{d}{4}$*),*

$$a \geq 0, \not\equiv 0 \quad \text{or} \quad \leq 0, \not\equiv 0 \quad \text{in } \Omega,$$

and

$$\exists k_0 \in \mathbb{N}, \sum_{j=1}^{m_{k_0}} (a, \varphi_{k_0 j})\varphi_{k_0 j}(x_0) \neq 0, \quad \lambda_{k_0} \neq 1.$$

Then $u_{\alpha,\beta}(x_0, t)$, $0 < t < T$

$\Longleftrightarrow (\alpha, \beta) \in (0, 2) \setminus \{1\} \times (0, 1)$ *1 to 1*

Simplified case.

Assume $m_k = 1$: λ_k is simple for all k, $-\Delta\varphi_k = \lambda_k\varphi_k$.

Corollary (uniqueness).

(i) $a \in H_0^2(\Omega)$ *if* $d = 1, 2, 3$,

$$a \geq 0, \not\equiv 0 \quad \textit{or} \quad \leq 0, \not\equiv 0 \quad \textit{in } \Omega$$

(ii) $\exists k_0 \in \mathbb{N}$, $(a, \varphi_{k_0})\varphi_{k_0}(x_0) \neq 0$ *and* $\lambda_{k_0} \neq 1$.

Then $u_{\alpha,\beta}(x_0, t)$, $0 < t < T \Longleftrightarrow$

$(\alpha, \beta) \in ((0, 2) \setminus \{1\}) \times (0, 1)$ *1 to 1*

Remark.

- (ii) is essential for uniqueness of β.

  Let $\lambda_1 = 1$ and $-\Delta\varphi_1 = \varphi_1$. Then

  $u_{\alpha,\beta}(x,t) = E_{\alpha,1}(-t^\alpha)\varphi_1(x)$ for $x \in \Omega,\ t > 0$.

  No information of β!
- (i) $\Longrightarrow$ uniqueness for α
- Tatar-Ulusoy (2013):

  $(a, \varphi_k) > 0$ for all $k \Longrightarrow$ uniqueness.

  Our proof producces the same conclusion.
- Let $\lambda_k \neq 1$ for all k.

  Then $a(x_0) \neq 0 \Longrightarrow$ uniqueness for β.

Main ingredients for proof.

(1) Eigenfunction expansion

(2) Asymptotics of Mittag-Leflller function

$$E_{\alpha,1}(-t) = \sum_{\ell=1}^{N} \frac{(-1)^{\ell+1}}{\Gamma(1-\alpha\ell)} \frac{1}{t^{\ell}} + O\left(\frac{1}{t^{1+N}}\right), \quad t > 0, \to \infty$$

(3) Strong maximum principle for $(-\Delta)^{\beta}$:

§3. Fractional diffusion equations backward in time for $0 < \alpha < 1$

with Profs G. Floridia (Università Mediterranea di Reggio Calabria) and Z. Li (Shandong University of Technology)

$-Au := \sum_{i,j=1}^{n} \partial_i(a_{ij}(x)\partial_j u) + c(x)u$,

where $c \le 0$, $\mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega)$.

$$\begin{cases} \partial_t^\alpha u = -Au \quad \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \quad u(\cdot, T) = b. \end{cases}$$

unique existence $u \in C([0,T];L^2(\Omega)) \cap C((0,T];H^2(\Omega) \cap H_0^1(\Omega))$ to $\partial_t^\alpha u = -Au$ with $u|_{\partial\Omega} = 0$ and $u(\cdot,T) = b$
(Sakamoto-Yamamoto 2011)

- memory effect or weak smoothing
- Different from $\alpha = 1$.

Our purpose:
Backward problem in more general cases
$\partial_t^\alpha u = -Au + F$?
Here $F = F(t)$? $F = F(t, u(t))$: semilinear

Ref:
N.H. Tuan - L.N. Huynh - T.B. Ngoc - Y. Zhou (2019),
We may get better results!?

Preliminaries

(I) $0 < \lambda_1 \le \lambda_2 \le \ldots.$: eignvalues of A with multiplicities,
φ_k: eigenfunction for λ_k, $(\varphi_j, \varphi_k) = \delta_{jk}$. $\|\cdot\| = \|\cdot\|_{L^2(\Omega)}$

(II) $E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}$ (Mittag-Leffler function).

(III) $S(t)a = \sum_{k=1}^{\infty}(a, \varphi_k)E_{\alpha,1}(-\lambda_k t^{\alpha})\varphi_k$,
$K(t)a = \sum_{k=1}^{\infty} t^{\alpha-1}(a, \varphi_k)E_{\alpha,\alpha}(-\lambda_k t^{\alpha})\varphi_k, \quad a \in L^2(\Omega)$

$\Longrightarrow$

$$\|S(t)a\|_{H^2(\Omega)} \le Ct^{-\alpha}\|a\|, \quad \|S(t)a\| \le C\|a\|$$

$$\|A^{\gamma}K(t)a\| \le Ct^{\alpha(1-\gamma)-1}\|a\|, \ 0 \le \gamma \le 1, \ t > 0$$

§3.2. Non self-adjoint A

$$-Lu := \sum_{i,j=1}^{n} \partial_i(a_{ij}(x)\partial_j u) + \sum_{j=1}^{n} b_j(x)\partial_j u(x) + c(x)u$$

$$=: -Au + Bu$$

$$\begin{cases} \partial_t^\alpha u = -Lu \quad \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \quad u(T) = b. \end{cases}$$

Theorem 3.

For $b \in H^2(\Omega) \cap H_0^1(\Omega)$, there exists unique solution u with same regularity as in Theorem 2.

Key to Proof

First Step.

$\partial_t^\alpha u = -Au + Bu$ in Ω, $u|_{\partial\Omega} = 0$, $u(\cdot, 0) = a \Longrightarrow$

$u_a(t) = S(t)a + \int_0^t K(t-s)Bu_a(s)ds$, where $S(t)$, $K(t)$ are constructed for $-A \Longrightarrow b = S(T)a + \int_0^T K(T-s)Bu_a(s)ds \Longleftrightarrow$

$$a = S(T)^{-1}\left(b - \int_0^T K(T-s)Bu_a(s)ds\right)$$

$$= S(T)^{-1}b - S(T)^{-1}\int_0^T K(T-s)Bu_a(s)ds = S(T)^{-1}b - Ma$$

Here

$$Ma := S(T)^{-1}\int_0^T K(T-s)Bu_a(s)ds$$

Unique solvability for a?

Second Step: compactness of $M : L^2(\Omega) \longrightarrow L^2(\Omega)$.

We can prove $\|Au(t)\| \le Ct^{-\alpha}\|a\|$ (Sakamoto-Yamamoto for $B = 0$)

Let $0 < \delta < \frac{1}{2}$.

$$\left\| A^{1+\delta} \int_0^T K(T-s)Bu_a(s)ds \right\| = \left\| \int_0^T A^{\frac{1}{2}+\delta} K(T-s) A^{\frac{1}{2}} Bu_a(s)ds \right\|$$

$$\le C \int_0^T (T-s)^{\alpha(\frac{1}{2}-\delta)-1} \|Au_a(s)\| ds \le C \int_0^T (T-s)^{\alpha(\frac{1}{2}-\delta)-1} s^{-\alpha} ds \|a\| \le C\|a\|.$$

$\Longrightarrow$

$$\left\| A^{\delta} S(T)^{-1} \int_0^T K(T-s)Bu_a(s)ds \right\| \le C \left\| A^{1+\delta} \int_0^T K(T-s)Bu_a(s)ds \right\| \le C\|a\|.$$

$\Longrightarrow M : L^2(\Omega) \longrightarrow H^{2\delta}(\Omega)$ is bounded

$M : L^2(\Omega) \longrightarrow L^2(\Omega)$ is compact

Third Step:

Fredholm equation of second kind:

$a = S(T)^{-1}b - Ma$

Well-posedness $\Longleftarrow$ "$b = 0 \Longrightarrow a = 0$"

$\Longleftrightarrow$ Backward uniqueness

$$\begin{cases} \partial_t^\alpha u = -Lu \quad \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \quad u(\cdot, T) = 0 \end{cases} \Longrightarrow \quad u(\cdot, 0) = 0.$$

Fourth Step: Backward uniqueness.

P_m, $m \in \mathbb{N}$: eigenprojection for eigenvalue λ_m of L

Closed subspace spanned by all the generalized eigenfunctions is $L^2(\Omega)$*.* (Agmon)

$\Longrightarrow$ $P_m a = 0$, $m \in \mathbb{N}$ imply $a = 0$.

Set

$$u_m(t) := P_m u(t), \quad a_m := P_m a$$

$\Longrightarrow$

$\partial_t^\alpha u_m = (-\lambda_m + D_m)u_m =: J u_m$ (Jordan form) where $D_m^\ell = 0$ with large ℓ.

$$\Longrightarrow \qquad u_m(t) = E_{\alpha,1}(Jt^\alpha)a_m = \sum_{k=0}^{\infty} \frac{(Jt^\alpha)^k}{\Gamma(\alpha k + 1)} a_m$$

Set $u_m = (u_m^1, \ldots, u_m^N)^T$ and $a_m = (a_m^1, \ldots, a_m^N)^T$ where ^{T}: tranpose
We can represent

$$u_m^1(t) = E_{\alpha,1}(-\lambda_m t^\alpha)a_m^1 + \cdots$$

$$u_m^2(t) = E_{\alpha,1}(-\lambda_m t^\alpha)a_m^2 + \cdots$$

$$\cdots\cdots$$

$$u_m^N(t) = E_{\alpha,1}(-\lambda_m t^\alpha)a_m^N$$

$u(T) = 0 \Longrightarrow u_m(T) = 0$

By $E_{\alpha,1}(-\lambda_m T^\alpha) \neq 0 \Longrightarrow$
$a_m^N = 0$, then
$a_m^{N-1} = 0$, then .....
$a_m^1 = 0$.
$\Longrightarrow a_m = 0$ for $m \in \mathbb{N}$.

Thus $a = 0$!

§4. Fractional wave equations backward in time for $1 < \alpha < 2$

with Profs G. Floridia (Università Mediterranea di Reggio Calabria)

Recall $-Av(x) = \sum_{i,j=1}^{n} \partial_i(a_{ij}(x)\partial_j v(x)) + c(x)v(x)$ with $c \le 0$, $\mathcal{D}(A) := H^2(\Omega) \cap H_0^1(\Omega)$.

Direct problem:

$$\begin{cases} \partial_t^\alpha u(x,t) = -Au(x,t), & x \in \Omega,\ t > 0, \\ u(\cdot,0) = a, \quad \partial_t u(\cdot,0) = b, & x \in \Omega. \end{cases}$$

$0 < \mu_1 < \mu_2 < \cdots \longrightarrow \infty$: set of eigenvalues
Let $\{\varphi_{nj}\}_{1\le j\le m_n}$: orthonormal basis of Ker $(A - \mu_n)$

Proposition (direct problem: Sakamoto-Yamamoto).
For $a, b \in L^2(\Omega)$, there exists a unique solution $u_{a,b}$:

$$\begin{cases} \lim_{t\to 0} \|u(\cdot,t) - a\|_{L^2(\Omega)} = \lim_{t\to 0} \|\partial_t u(\cdot,t) - b\|_{H^{-2}(\Omega)} = 0, \\ u_{a,b}(x,t) = \sum_{n=1}^{\infty} \sum_{j=1}^{m_n} \{(a,\varphi_{nj}) E_{\alpha,1}(-\mu_n t^\alpha) \\ +(b,\varphi_{nj}) t E_{\alpha,2}(-\mu_n t^\alpha)\} \varphi_{nj}(x) \qquad \text{in } C([0,T]; L^2(\Omega)), \\ \partial_t u_{a,b}(x,t) = \sum_{n=1}^{\infty} \sum_{j=1}^{m_n} \{-\mu_n t^{\alpha-1} (a,\varphi_{nj}) E_{\alpha,\alpha}(-\mu_n t^\alpha) \\ +(b,\varphi_{nj}) E_{\alpha,1}(-\mu_n t^\alpha)\} \varphi_{nj}(x) \qquad \text{in } C([0,T]; H^{-2}(\Omega)). \end{cases}$$

Backward problem:

Let $T > 0$ and a_T, b_T be given. Find $u = u(x, t)$ such that

$$\begin{cases} \partial_t^\alpha u = -Au, \quad x \in \Omega,\ t > 0, \\ u(\cdot, T) = a_T, \quad \partial_t u(\cdot, T) = b_T, \qquad x \in \Omega \end{cases}$$

Theorem 4.1 (generic well-posedness).

There exists a finite set $\{\eta_1, ..., \eta_N\} \subset (0, \infty)$ *satisfying:*

(i) If

$$T \notin \bigcup_{n=1}^{\infty} \left\{ \left(\frac{\eta_1}{\mu_n}\right)^{\frac{1}{\alpha}}, \cdots, \left(\frac{\eta_N}{\mu_n}\right)^{\frac{1}{\alpha}} \right\},$$

then for any $a_T, b_T \in H^2(\Omega) \cap H_0^1(\Omega)$, *there exists a unique* $a, b \in L^2(\Omega)$ *such that* $u_{a,b}(\cdot, T) = a_T$ *and* $\partial_t u_{a,b}(\cdot, T) = b_T$ *and* $||a_T||_{H^2(\Omega)} + ||b_T||_{H^2(\Omega)} \sim ||a||_{L^2(\Omega)} + ||b||_{L^2(\Omega)}$.

(ii) No uniqueness for backward problem if

$$T \in \bigcup_{n=1}^{\infty} \left\{ \left(\frac{\eta_1}{\mu_n}\right)^{\frac{1}{\alpha}}, \cdots, \left(\frac{\eta_N}{\mu_n}\right)^{\frac{1}{\alpha}} \right\}.$$

Remark:

The set $\bigcup_{n=1}^{\infty}\left\{\left(\frac{\eta_1}{\mu_n}\right)^{\frac{1}{\alpha}},\cdots,\left(\frac{\eta_N}{\mu_n}\right)^{\frac{1}{\alpha}}\right\}$

- has $\mathbf{0}$ as unique accumulation point
- is included in $\left[0,\left(\frac{\eta_N}{\mu_1}\right)^{\frac{1}{\alpha}}\right] \Longrightarrow$

backward problem is well-posed for $T > \left(\frac{\eta_N}{\mu_1}\right)^{\frac{1}{\alpha}}$

Summary for backward problem in time.

- $0 < \alpha < 1$: well-posed for any $T > 0$.
- $\alpha = 1$: severely ill-posed but uniqueness and conditional stability for any $T > 0$.
- $1 < \alpha < 2$: well-posed for $T > 0$ not belonging to a countably infinite set. Non-uniqueness for such exceptional values of T.
- $\alpha = 2$: Well-posed. Also conservation quantity such as energy, which is impossible for $\alpha \neq 2$.

Sketch of Proof. $u(\cdot,T)=a_T$ and $\partial_t u(\cdot,T)=b_T \Longleftrightarrow$

$$\begin{cases} (a,\varphi_{nj})E_{\alpha,1}(-\mu_n T^\alpha)+(b,\varphi_{nj})TE_{\alpha,2}(-\mu_n T^\alpha)=(a_T,\varphi_{nj}), \\ (a,\varphi_{nj})(-\mu_n T^{\alpha-1}E_{\alpha,\alpha}(-\mu_n T^\alpha))+(b,\varphi_{nj})E_{\alpha,1}(-\mu_n T^\alpha)=(b_T,\varphi_{nj}) \end{cases}$$

$\Longrightarrow$ linear system w.r.t. (a,φ_{nj}) and (b,φ_{nj})

Determinant of coefficient matrix $=\psi(\mu_n T^\alpha)$, where

$$\psi(\eta):=E_{\alpha,1}(-\eta)^2+\eta E_{\alpha,2}(-\eta)E_{\alpha,\alpha}(-\eta),\quad \eta>0$$

Well-posed of backward problem $\Longleftrightarrow \psi(\mu_n T^\alpha)\neq 0$, $\forall n$

Asymptotics of Mittag-Leffler functions $\Longrightarrow \psi(\infty)<0$
$\psi(0)=1>0$ Intermediate value theorem $\Longrightarrow \exists\eta_0\ \psi(\eta_0)=0$

η: analytic in $\eta>0 \Longrightarrow$ finite zeros: $\eta_1,\ldots,\eta_N\in(0,\infty)$

$\eta(\eta_k)=0 \Longleftrightarrow \mu_n T^\alpha=\eta_k$, that is, $T=\left(\frac{\eta_k}{\mu_n}\right)^{\frac{1}{\alpha}}$

Thank you very much.